Cycle multiplicity of total graph of $C_n$, $P_n$, and $K_{1,n}$

M.M. Akbar Ali$^{1*}$, S. Panayappan$^2$

$^1$Department of Mathematics, Sri Shakthi Institute of Engineering and Technology, Coimbatore, Tamil Nadu, INDIA
$^2$Department of Mathematics, Government Arts College, Coimbatore, Tamil Nadu, INDIA

*Corresponding Author: e-mail: um_akbar@yahoo.co.in, Tel +91-9994279835.

Abstract

Cycle multiplicity of a graph $G$ is the maximum number of edge disjoint cycles in $G$. In this paper, we find the cycle multiplicity of total graph of cycles $C_n$, paths $P_n$ and star graph $K_{1,n}$ respectively.

Keywords: cycle multiplicity, total graph, cycle, path, star graph.

1. Introduction

Line partition number (Chatrand et al., 1971) of a graph $G$ is the minimum number of subsets into which the edge-set of $G$ can be partitioned so that the subgraph induced by each subset has property $P$. Dual to this concept of line partition number of graph is the maximum number of subsets into which the edge-set of $G$ can be partitioned such that the subgraph induced by each subset does not have the property $P$. Define the property $P$ such that a graph $G$ has the property $P$ if $G$ contains no subgraph which is homeomorphic from the complete graph $K_3$. Now the line partition number and dual line partition number corresponding to the property $P$ is referred to as arboricity and cycle multiplicity of $G$ respectively. Equivalently the cycle multiplicity is the maximum number of line disjoint subgraphs contained in $G$ so that each subgraph is not acyclic. This number is called the cycle multiplicity of $G$ denoted by $CM(G)$. The formula for cycle multiplicity of a complete and complete bipartite graph is given in (Chatrand et al., 1971). In (Simões Pereira, 1972), the author found an upper bound for the line and middle graph of any graph. Also he proved that the bound becomes the formula for line and total graph of any forest.

We consider finite, simple, undirected graph $G(V(G), E(G))$ where $V(G)$ and $E(G)$ represent vertex set and edge set of $G$ respectively. For any real number $r$, $[r]$ and $\lceil r \rceil$ denote the largest integer not exceeding $r$ and the least integer not less than $r$, respectively. The other notations and terminology used in this paper can be found in (Harary, 1969).

1.2 Cycle multiplicity of total graph of $C_n$

It is obvious that cycle multiplicity of any cycle is one. We obtain a formula to find the cycle multiplicity of the total graph of a cycle.

Theorem 2.1

Cycle multiplicity of Total Graph of $n$-Cycle, $CM[T(C_n)] = n + 1$
Proof

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$ in which $e_i = v_iv_{i+1}$. By the definition of total graph, $V(T(C_n)) = \{v_1, v_2, ..., v_n\} \cup \{e_1, e_2, ..., e_n\}$ and $E(T(C_n)) = \{e_1e_2 / (1 \leq i \leq n - 1)\} \cup \{v_iv_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_nv_1\}$. The cycles of $T(C_n)$ are $C_i = v_iv_{i+1}e_i (1 \leq i \leq n - 1)$, $C_n = v_nv_1$. Let $C_{n+1} = v_1v_2....v_nv_1$, $C_{n+2} = e_1e_2....e_ne_1$, $C_{n+3} = v_1e_1v_2e_2v_3....v_nv_1$.

Now we collect set of line disjoint cycles, $\mathcal{C}_i = \{C_i / (1 \leq i \leq n - 1)\} \cup \{C_n\}$ and $\mathcal{C}_2 = \{C_n\} \cup \{C_{n+1}\} \cup \{C_{n+2}\} \cup \{C_{n+3}\}$. Clearly $\mathcal{C}_i (1 \leq i \leq 3)$ is a set of line disjoint cycles in $T(C_n)$ and $|\mathcal{C}_i| = |\mathcal{C}_2| = n + 1$. Since $n \geq 3$, $|\mathcal{C}_i| \geq |\mathcal{C}_2|$ and either $\mathcal{C}_i$ or $\mathcal{C}_2$ contains maximum number of line disjoint cycles of $T(C_n)$ and hence $CM[T(C_n)] = n + 1$.

3. Cycle multiplicity of total graph of $P_n$

As $P_n$ does not contain any cycle, its cycle multiplicity is zero. In the following theorem we state a formula to find the maximum number of line disjoint cycles in the total graph of a path.

Theorem 3.1

Cycle multiplicity of total graph of path, $CM[T(P_n)] = n$

Proof

Let $V(P_n) = \{v_1, v_2, ..., v_{n+1}\}$ and $E(P_n) = \{e_1, e_2, ..., e_n\}$. By the definition of total graph, $V(T(P_n)) = V(P_n) \cup E(P_n)$, $E(T(P_n)) = \{v_ie_i / (1 \leq i \leq n)\} \cup \{e_{i+1}i / 1 \leq i \leq n\} \cup \{v_iv_{i+1} / 1 \leq i \leq n\}$, $\cup \{e_{i+1}i / 1 \leq i \leq n - 1\}$ The cycles of $T(P_n)$ are
Ci = viVi+1ei (1 ≤ i ≤ n) and C1 = eiVi+1ei (1 ≤ i ≤ n - 1). Let C1 = {Ci | (1 ≤ i ≤ n)} and C2 = {C1 | 1 ≤ i ≤ n - 1}. The cycles in the set C1 (i = 1, 2) are line disjoint cycles of T(Pn). Also |C1| < |C2| = n and hence CM[T(Pn)] = n

4. Cycle multiplicity of total graph of K1,n

Since the star graphs are acyclic its cycle multiplicity is zero. We find a formula for the cycle multiplicity of total graph of a star graph

Theorem 4.1

Cycle multiplicity of total graph of Km,n, CM[T(K1,n)] = \[
\left\{ \begin{array}{ll}
\frac{n^2 + 5n}{6} & \text{if } n \text{ is odd} \\
\frac{n(n+4)}{6} & \text{if } n \text{ is even}
\end{array} \right.
\]

Proof

Let V(K1,n) = {v1, v2, ..., vn} and E(K1,n) = {e1, e2, ..., en}. By the definition of total graph, we have V(T(K1,n)) = {v1} ∪ {ei / (1 ≤ i ≤ n)} ∪ {vi / (1 ≤ i ≤ n)}, in which the vertices e1, e2, ..., en induces a cliques of order n (say Kn). Also the vertex v is adjacent with vi (1 ≤ i ≤ n).

Case (i)

If n is odd

We collect the set of line disjoint cycles of T(K1,n) as below.

\[\mathcal{C}_1 = \{ve_i, e_i v / (i = 1, 2, ..., n-2)\}, \quad \mathcal{C}_2 = \{ve_i, e_i v / (1 ≤ i ≤ n)\}, \quad \mathcal{C}_3 = \{\text{set of line disjoint cycles in the clique } Kn\}.

\[\mathcal{C}_4 = \{ve_i v / (1 ≤ i ≤ n)\}, \quad \text{Clearly } |\mathcal{C}_1| = |\mathcal{C}_3| = \frac{n-1}{2}.

To prove |\mathcal{C}_1| = \left[\frac{n^2 - n}{6}\right], \text{ i.e., we have to prove the number of line disjoint cycles in } \mathcal{C}_1 \text{ is } \left[\frac{n^2 - n}{6}\right] \text{ if } n \text{ is odd. If } n = 1, \text{ the number of line disjoint cycles in the clique } K_1 \text{ is zero and } \left[\frac{n^2 - n}{6}\right] = 0 \text{ for } n = 1. \text{ Similarly if } n = 3, \text{ then the number of line
disjoint cycles in the clique $K_3$ is one and $\frac{n^2 - n}{6} = 1$ for $n = 2$. Therefore $|\mathcal{C}_3| = \frac{n^2 - n}{6}$ if $n = 1, 3$. Assume that the result is true for $m = 2k - 1$ for some $k$. i.e., $|\mathcal{C}_3| = \frac{2k^2 - 3k + 1}{3}$, i.e., Number of line disjoint cycles in $K_m = |\mathcal{C}_3| = \frac{2k^2 - 3k + 1}{3}$.

Now consider the clique $K_n$ where $n = 2k + 1$. Consider $K_{n-2} = K_n - \{e_{2k}, e_{2k+1}\} = K_{2k-1} = K_m$. Number of line disjoint cycles in $K_{n-2}$ is $\frac{2k^2 - 3k + 1}{3} + \frac{4k - 1}{3}$. Therefore $|\mathcal{C}_3|$ in $K_n$ is $\frac{2k^2 - 3k + 1}{3} + \frac{4k - 1}{3}$. i.e., $|\mathcal{C}_3| = \frac{2k^2 + k}{3}$, i.e., $|\mathcal{C}_3| = \frac{n^2 - n}{6}$, where $n = 2k + 1$. Since $n$ is odd there exist no edges in the clique which are left out in the extraction of line disjoint cycles. Since $n \geq 2$, $\frac{n - 2}{2} \leq \frac{n^2 - n}{6}$. Therefore $|\mathcal{C}_3| = |\mathcal{C}_3|$. The cycles in $\mathcal{C}_3$ and $\mathcal{C}_4$ are line disjoint. Therefore maximum number of line disjoint cycles in $T(K_n, d)$, $CM[T(K_n, d)] = |\mathcal{C}_3| + |\mathcal{C}_4| = \frac{n^2 - n}{6} + n = \frac{n^2 + 5n}{6}$ if $n$ is odd.

Case (ii)

If $n$ is even

In this case we collect the set of line disjoint cycles as below.

$\mathcal{C}_1 = \{ve_i, v_i, v/ i = 1, 3, \ldots, n-1\}$, $\mathcal{C}_2 = \{ve_i, v_i, v/ i = 2, 4, \ldots, n-2\}$, clearly $|\mathcal{C}_3| = \frac{n}{2}$ and $|\mathcal{C}_3| = \frac{n^2 - n}{6}$. $\mathcal{C}_3 = \{set of line disjoint cycles in $K_n\}$. $\mathcal{C}_4 = \{ve_{i, v}/ (1 \leq i \leq n) \}$, We prove $|\mathcal{C}_3| = \frac{n(n - 2)}{6}$. Maximum number of line disjoint cycles are extracted from $K_n$ using the following steps.

Step 1:

Extract the line disjoint cycles $c_i = e_ie_{i+1}e_{i+2}e_i (i = 1, 3, \ldots, n-1)$. Clearly $c_1, c_2, \ldots, c_{n-1}$ are line disjoint cycles.

Thus we get $\frac{n}{2}$ line disjoint cycles.

Step 2:

Delete the edges $e_i e_{\left\lfloor \frac{n}{2} \right\rfloor} (i = 1, 2, \ldots, \frac{n}{2})$ from $K_n$.

Step 3:

Extract $\frac{n^2 - 5n}{6}$ line disjoint 3-cycles from $K_n - \{e_i e_{\left\lfloor \frac{n}{2} \right\rfloor} (i = 1, 2, \ldots, \frac{n}{2}) \}$. 


Therefore \(\frac{n}{2} + \frac{n^2 - 5n}{6} = \left[\frac{n(n - 2)}{6}\right]\). Since \(e_i e_j e_{\frac{n}{2} + i}\) \((i = 1, 2, \ldots, \frac{n}{2})\) are mutually non adjacent edges in \(T(K_{1, n})\). Let \(C_3\)

\[\{v e_i e_{\frac{n}{2} + i}/ (i = 1, 2, \ldots, \frac{n}{2})\}\]. The cycles in \(C_3\) are line disjoint. The cycles in \(C_3\) and \(C_5\) are line disjoint and also the cycles in \(C_3\) and \(C_4\) are line disjoint. Since \(|C_5| \leq |C_4|\). Therefore maximum number of line disjoint cycles in \(T(K_{1, n})\),

\[CM[T(K_{1, n})] = |C_3| + |C_4| = \left[\frac{n(n - 2)}{6}\right] + n = \left[\frac{n(n + 4)}{6}\right]\].

Acknowledgement

The present version of this paper owes much to the referees for their comments, suggestions and remarks that have resulted in the improvement of this paper.

References

Michalak D., 1981. *On middle and total graphs with coarseness number equal 1*, Springer Verlag Graph Theory, Lagow proceedings, Berlin Heidelberg, New York, Tokyo, 139 -150.

Biographical notes

**M.M. Akbar Ali** is working as an Assistant Professor in the Department of Mathematics, Sri Shakthi Institute of Engineering and Technology, Coimbatore, India. He received his M.Phil degree in Graph Theory from Bharathiar University in 2004 and his M.Sc from Government Arts College, Coimbatore and currently pursuing PhD from Bharathiar University, Coimbatore. He has Published 16 papers in National and International Journals/Proceedings. He has participated in 15 National and International Conferences and presented more than 10 papers. Won the Best Paper Award at the International Conference on Mathematics and Computer Science (ICMCS 2007), March 1-3, 2007, Loyola College, Chennai, India. Area of interest includes Graph Colorings and Decomposition of graphs.

**Dr. S. Panayappan** has 25 years of research experience and completed 5 research projects. He is presently serving as Principal investigator of University Grant Commission’s Major project at Government Arts College Coimbatore. 45 M.Phil and 5 PhD scholars have been awarded the degree under his supervision and he is guiding 8 PhD scholars at present. He has Published 25 papers in National and International Journals/Proceedings. He has organized 3 conferences, including one graph theory conference. Area of interest includes Graph theory and Operator Theory.

Received February 2010
Accepted March 2010
Final acceptance in revised form April 2010